8.2

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hu) = 0 & (1) \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + g\dfrac{\partial h}{\partial x} = 0 & (2) \end{cases}$$

$$\vec{X} = \begin{pmatrix} h \\ u \end{pmatrix}$$

$$\frac{\partial \vec{X}}{\partial t} + \underline{\underline{A}}\,\frac{\partial \vec{X}}{\partial x} = 0 \quad \text{with } \underline{\underline{A}} = \begin{bmatrix} u & h \\ g & u \end{bmatrix}$$

<u>eigenvalues of $\underline{\underline{A}}$</u>

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

$$\Leftrightarrow (u-\lambda)^2 - gh = 0$$

$$\Leftrightarrow (u-\lambda)^2 = gh$$

$$\Leftrightarrow \lambda = u \pm \sqrt{gh}$$

We set $c = \sqrt{gh}$ (celerity of free surface waves)

Then $\lambda = u \pm c$

information speed = mean flow $\pm$ wave speed

<u>Left eigenvectors</u>

We have to solve

$$\vec{v}.\underline{\underline{A}} = \lambda \vec{v}$$

we set $\vec{v} = \begin{pmatrix} 1 \\ y \end{pmatrix}$ see slide 29

$$\begin{cases} u + g\,y = \lambda \\ h + u\,y = \lambda y \end{cases}$$

$$\Rightarrow \quad y = \frac{\lambda - u}{g}$$

Since $\lambda = u \pm c$, then $\quad y = \pm \frac{c}{g}$

We set $\quad \lambda_1 = u - c \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ -\frac{c}{g} \end{pmatrix}$

$\qquad\qquad \lambda_2 = u + c \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ \frac{c}{g} \end{pmatrix}$

<u>Diagonalization</u>

To uncouple equations (1) and (2) we have to introduce the Riemann variables r_1 and r_2 such that

$$\vec{v}\cdot\frac{d\vec{X}}{dt} = \vec{v}\left(\frac{\partial \vec{X}}{\partial t} + \underline{A}\,\frac{\partial \vec{X}}{\partial x}\right) = \mu\,\frac{dr}{dt}$$

We seek the integrating factor μ and the Riemann variables r:

$$\vec{v}\,.\,d\vec{X} = \mu\,\partial r = \mu\left(\frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial x}dx\right)$$

$$\Downarrow$$

$$dh \pm \frac{c}{g}\,du$$

$$\mu \frac{\partial r}{\partial h} = 1$$

$$\mu \frac{\partial r}{\partial u} = \pm \frac{c}{g}$$

Taking the ratio, we get

$$\frac{\partial r}{\partial h} = \pm \frac{g}{c} \frac{\partial r}{\partial u}$$

- Let us start with r_1 associated with $\lambda_1 = u - c$

$$\frac{\partial r}{\partial h} = - \frac{g}{c} \frac{\partial r}{\partial u}$$

integrals of the form

$$\frac{dh}{1} = \frac{du}{\frac{g}{c}} = \frac{dr}{0}$$

we solve $\quad du = \frac{g}{c} dh = \sqrt{\frac{g}{h}}\, dh$

$$u = 2\sqrt{gh}$$

we pose $\quad \boxed{r_1 = u - 2c}$

- Similarly, for r_2 associated with $\lambda_2 = u + c$

$$\frac{\partial r}{\partial h} = + \frac{g}{c} \frac{\partial r}{\partial u}$$

$$\Rightarrow \boxed{r_2 = u + 2c}$$

<u>Summary</u> Equations (1) and (2) are equivalent to

$$\begin{cases} \dfrac{dr_1}{dt} = 0 & \text{along } \dfrac{dx}{dt} = \lambda_1 \\ \dfrac{dr_2}{dt} = 0 & \dfrac{dx}{dt} = \lambda_2 \end{cases}$$

Determination of the "rarefaction wave" solution

(1) and (2) are invariant to $x \to \lambda x$

$t \to \lambda t$

We pose $\xi = \frac{x}{t}$ and seek similarity solutions in the form

$$X = \begin{pmatrix} h \\ \mu \end{pmatrix} = \vec{W}(\xi)$$

$$\frac{\partial \vec{X}}{\partial t} + \underline{\underline{A}} \frac{\partial \vec{X}}{\partial x} = 0 \qquad (3)$$

$$\Rightarrow \frac{\partial X}{\partial t} = \frac{\partial \xi}{\partial t} W' = -\frac{\xi}{t} W'$$

$$\frac{\partial X}{\partial x} = \frac{\partial \xi}{\partial x} W' = +\frac{1}{t} W'$$

Thus (3) $\Longleftrightarrow -\xi \vec{W}' + \underline{\underline{A}}\, \vec{W}' = 0$

This means that $\vec{W}'$ is a right eigenvector of $\underline{\underline{A}}$

See slide 44

$$\xi = \lambda(W)$$

Differentiation lead to $1 = \nabla\lambda \cdot \vec{W}'$

and since $\vec{W}'$ is a right eigen vector of $\underline{\underline{A}}$, there is α such that $\vec{W}' = \alpha \vec{w}$

We then have $\alpha = \nabla\lambda \cdot \vec{w}$

and thus $\boxed{\vec{W}' = \frac{\vec{w}}{\nabla\lambda \cdot \vec{w}}} \qquad (4)$

8.2

For the shallow water equations, the right eigenvectors are

$$\vec{w}_1' = \begin{pmatrix} -\frac{c}{g} \\ 1 \end{pmatrix} \text{ and } \vec{w}_2' = \begin{pmatrix} \frac{c}{g} \\ 1 \end{pmatrix}$$

(<u>Note</u> $\vec{w}_1' \cdot \vec{v}_2 = 0$ and $\vec{w}_2' \cdot \vec{v}_1 = 0$)

Here

- For $\lambda_1 = u - c$

$$\nabla \lambda_1 = \left(\frac{\partial \lambda_1}{\partial h}, \frac{\partial \lambda}{\partial u} \right) = \left(\frac{1}{2} \frac{c}{h}, 1 \right)$$

$$\nabla \lambda_1 \cdot \vec{w}_1' = \frac{c^2}{2gh} + 1 = \frac{3}{2}$$

$$(4) \Rightarrow \vec{w}_1' = \frac{2}{3} \begin{pmatrix} -\frac{c}{g} \\ 1 \end{pmatrix}$$

$$\begin{cases} \frac{dh}{d\xi} = -\frac{2}{3} \sqrt{\frac{h}{g}} \quad \Rightarrow \quad \frac{dh}{\sqrt{h}} = -\frac{2}{3} \frac{d\xi}{\sqrt{g}} \\ \frac{du}{d\xi} = \frac{2}{3} \qquad\qquad du = \frac{2}{3} d\xi \end{cases}$$

The solution is

$$\begin{cases} h = \frac{1}{9g} (cst - \xi)^2 & (5) \\ u = \frac{2}{3} \xi + cst & (6) \end{cases}$$

- For $\lambda_2 = u + c$

$$\nabla \lambda_2 = \left(\frac{1}{2} \frac{c}{h} , 1 \right)$$

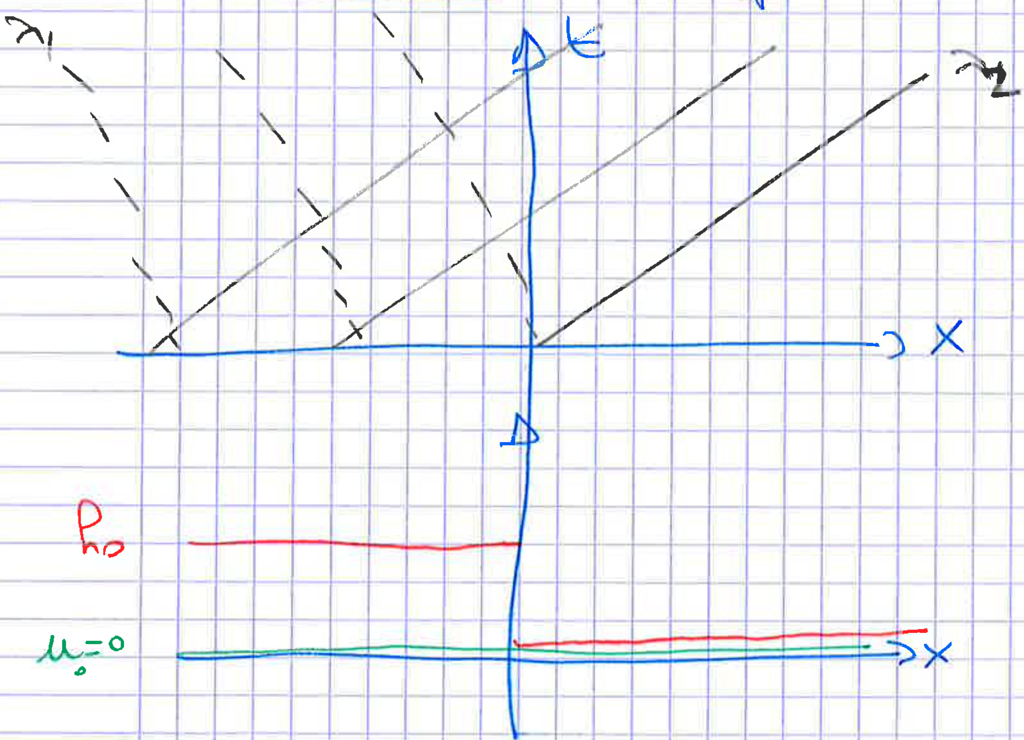
$$\nabla \lambda_2 \cdot \vec{w}_2' = \frac{3}{2}$$

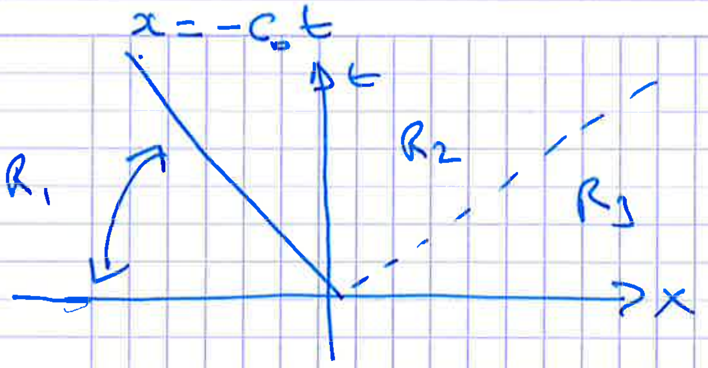
$$\vec{w}_2' = \frac{2}{3} \begin{pmatrix} \frac{c}{g} \\ 1 \end{pmatrix}$$

$$h = \frac{1}{9g} \left(cst + \xi \right)^2$$

$$u = \frac{2}{3} \xi$$

Solution to the Riemann problem:



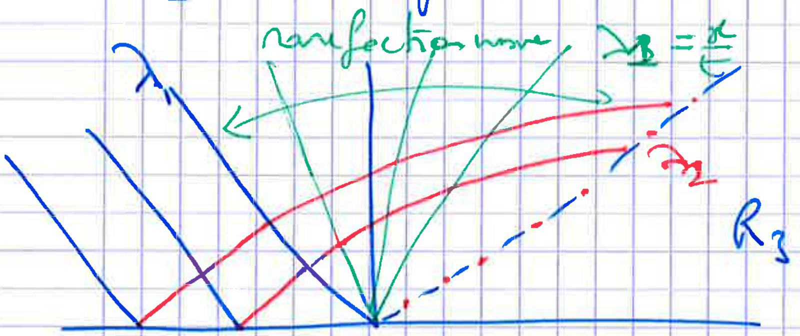


$c_0 = \sqrt{g h_0}$

R_1 = undisturbed domain $\begin{cases} u=0 \\ h=h_0 \end{cases}$

R_3 = vacuum $h=u=0$

R_2 = rarefaction wave solution.



we have in the R_2 domain

λ_1 characteristics : $\frac{x}{t} = u-c$

λ_2 characteristics $\frac{dr_2}{dt} = 0$ along $\frac{dx}{dt} = \lambda_2$

if we take any point in R_2, two characteristics cross eachother there

$$\begin{cases} \frac{x}{t} = u-c \\ r_2 = r_2(t=0) = \cancel{u_0}^{0} + 2c_0 = 2c_0 \Leftrightarrow u+2c = 2c_0 \end{cases}$$

$$\begin{cases} u + 2c = 2c_0 \\ u - c = \dfrac{x}{t} \end{cases}$$

$$\Rightarrow \quad 3c = 2c_0 - \frac{x}{t}$$

$$\Rightarrow \quad c = \left(2c_0 - \frac{x}{t}\right)\frac{1}{3}$$

$$\Rightarrow \quad h = \frac{1}{9g}\left(2c_0 - \frac{x}{t}\right)^2$$

and $u = \dfrac{x}{t} + c = \dfrac{2c_0}{3} + \dfrac{2}{3}\dfrac{x}{t}$

$$= \frac{2}{3}\left(c_0 + \frac{x}{t}\right)$$

The solution can also be obtained from Eqs (5) and (6). The constant of integration are determined

$h = 0$ at $x = x_f$ $\quad \dot{x}_f = u_f$

$h = h_0$, $u = 0$ at $x = x_b$

for the back point:

$$(5) \Rightarrow \frac{1}{9g}\left(a-\xi_b\right)^2 = h_0$$

$$(6) \Rightarrow \frac{2}{3}\xi_b + b = 0$$

and for the front:

$$(5) \quad \frac{1}{9g}\left(a-\xi_f\right)^2 = 0$$

$$(6) \Rightarrow \xi_f = \frac{2}{3}\xi_f + b$$

Solving for a, b, ξ_b, ξ_f gives

$$\begin{cases} a = 2c_0 \\ b = \frac{2}{3}c_0 \\ \xi_f = 2c_0 \\ \xi_b = -c_0 \end{cases}$$

Thus $\begin{cases} h = \dfrac{1}{9g}\left(2c_0 - \dfrac{x}{t}\right)^2 \\ u = \dfrac{2}{3}\left(c_0 + \dfrac{x}{t}\right) \end{cases}$

8.3

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

$$\Leftrightarrow \left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$$

$$\Leftrightarrow \begin{cases} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0 \end{cases}$$

$$\frac{dt}{1} = \pm\frac{dx}{c} = \frac{du}{0}$$

Solutions : $u = F(x \pm ct)$

D'Alembert solution $u = F(x+ct) + G(x-ct)$

Boundary conditions

$$\begin{cases} u(x,0) = f(x) & (2) \\ \frac{\partial u}{\partial t}(x,0) = g(x) & (3) \end{cases}$$

Solutions

$(3) \Leftrightarrow \quad F'(x) - G'(x) = \frac{1}{c} g(x)$

$(2) \Leftrightarrow \quad F(x) + G(x) = f(x)$

Differentiating the last equation.

$$F' + G' = f'$$

So

$$2F'(x) = \frac{1}{c} g + f'$$

$$F(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(x')\,dx'$$

$$G(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(x')\,dx'$$

We deduce:

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$= \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(x')\,dx'$$