Homework

$$\frac{\partial h}{\partial t} + \kappa \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) = 0$$

BC $\begin{cases} \int_0^{x_f} h\, dx = V_0 \\ h(x_f) = 0 \end{cases}$ $\kappa = \frac{g}{3\nu}$

Scaling
$x \to \lambda x$
$t \to \lambda^a t$
$h \to \lambda^b h$

$$\lambda^{b-a} \partial_t h + \kappa \lambda^{4b-2} \partial_x (h^3 \partial_x h) = 0$$

$$b - a = 4b - 2 \Rightarrow 3b = 2 - a$$

$$\lambda^{b+1} \int_0^{x_f} h\, dx = V_0 \Rightarrow b = -1$$

$$a = 5$$

$$\frac{dx}{x} = \frac{dt}{5t} = -\frac{dh}{h}$$

$$\Rightarrow h = H(\xi)\, t^{-\frac{1}{5}} \qquad \xi = \frac{x}{t^{1/5}}$$

$$+H + \xi H' + 5\kappa(3 H^2 H'^2 + H^3 H'') = 0$$

By integration

$$\xi H + 5\kappa H^3 H' = a$$

at $\xi = \xi_f \quad H = 0 \quad \Rightarrow \quad e = 0$

$$H^2 H' = -\frac{\xi}{5K}$$

$$\frac{1}{3} H^3 = -\frac{\xi^2}{10K} + a$$

at $\xi = \xi_f \quad H = 0 \quad \Rightarrow \quad a = \frac{\xi_f^2}{10K}$

$$H = \left[\frac{3}{K}\,\frac{\xi_f^2 - \xi^2}{10}\right]^{1/3}$$

$$h = t^{-1/5} H(\xi)$$

ξ_f is fixed by the BC: $\int_0^{x_f} h\,dx = V_0$

$$\int_0^{\xi_f} H\,d\xi = V_0$$

$$H = \frac{\xi_f^{2/3}}{K^{1/3}}\,\phi(\eta) \qquad \phi = \left[\frac{3}{10}(1-\eta^2)\right]^{1/3} \qquad \eta = \frac{\xi}{\xi_f}$$

$$\frac{\xi_f^{5/3}}{K^{1/3}} \int_0^1 \phi(\eta)\,d\eta = V_0$$

$$\xi_f = K^{1/5}\, V_0^{3/5}\, \alpha$$

$$\alpha = \frac{1}{\left[\int_0^1 \phi(\eta)\,d\eta\right]^{3/5}} = \left[\frac{5\left(\frac{10}{3}\right)^{1/3}\Gamma\left(\frac{5}{6}\right)}{\sqrt{\pi}\;\Gamma\left(\frac{1}{3}\right)}\right]^{3/5} \sim 1.41$$