Chapter 4: First-order differential equations

Similarity and Transport Phenomena in Fluid Dynamics

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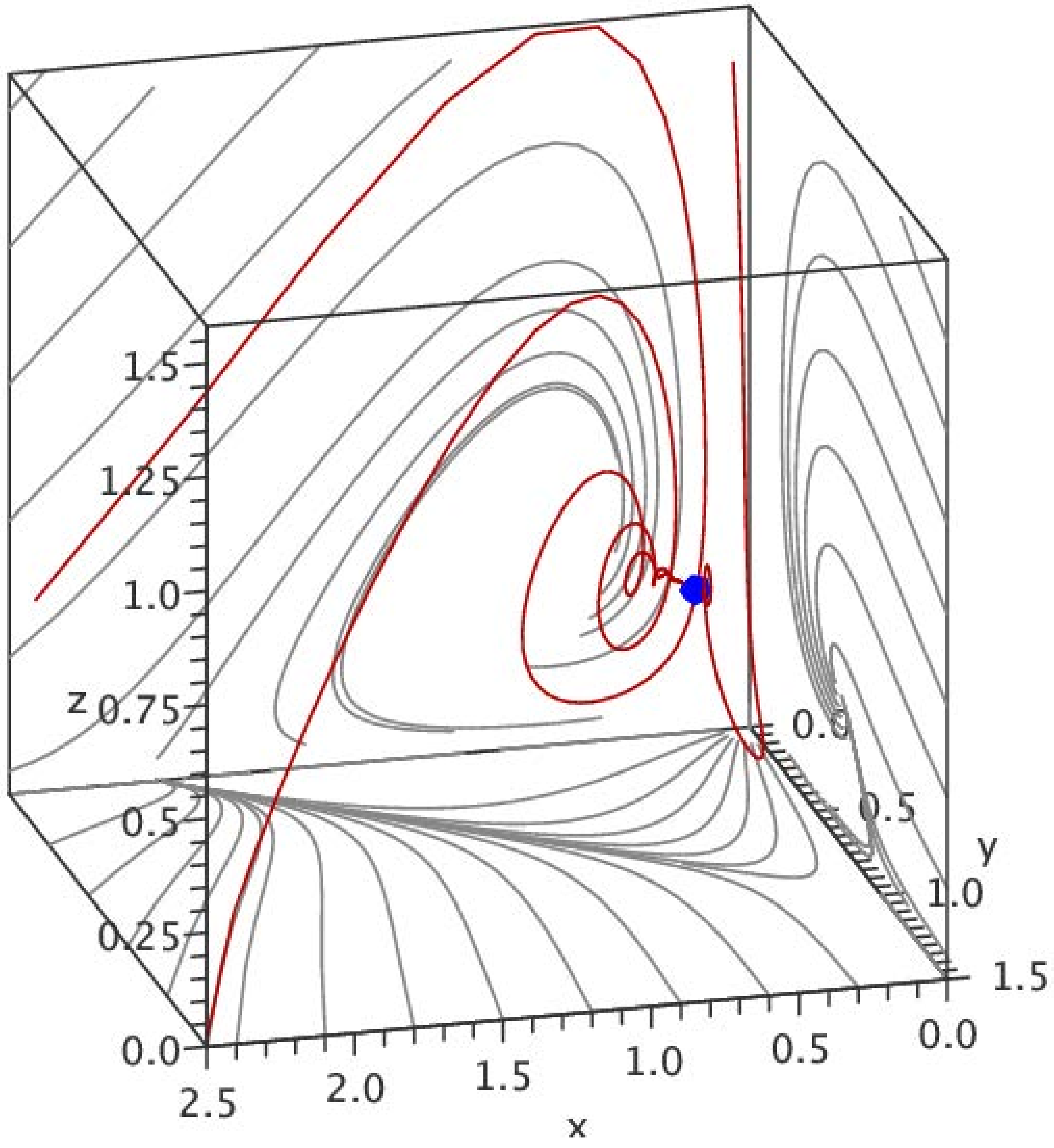
Chapter 4: First-order differential equations



- Phase portrait
- Singular point
- Separatrix
- Integrating factor
- Invariant integral curves
- Singular solution
- Change of variable

Phase portrait

A number of nonlinear first-order equations as well as second-order autonomous equations can be cast in the following form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x,\ y)}{g(x,\ y)},$$

with f and g two functions that may vanish.

The points that are both zeros of f and g are called *singular points* (*critical* or *equilibrium* points). The behaviour of the integral curves depends strongly on the structure of curves $f(x,\ y) = 0$ and $g(x,\ y) = 0$ around these critical points, i.e. the multiplicity of critical curves generated by the equations $f(x,\ y) = 0$ and $g(x,\ y) = 0$ and by the sign of f/g in the different areas delineated by these critical curves.

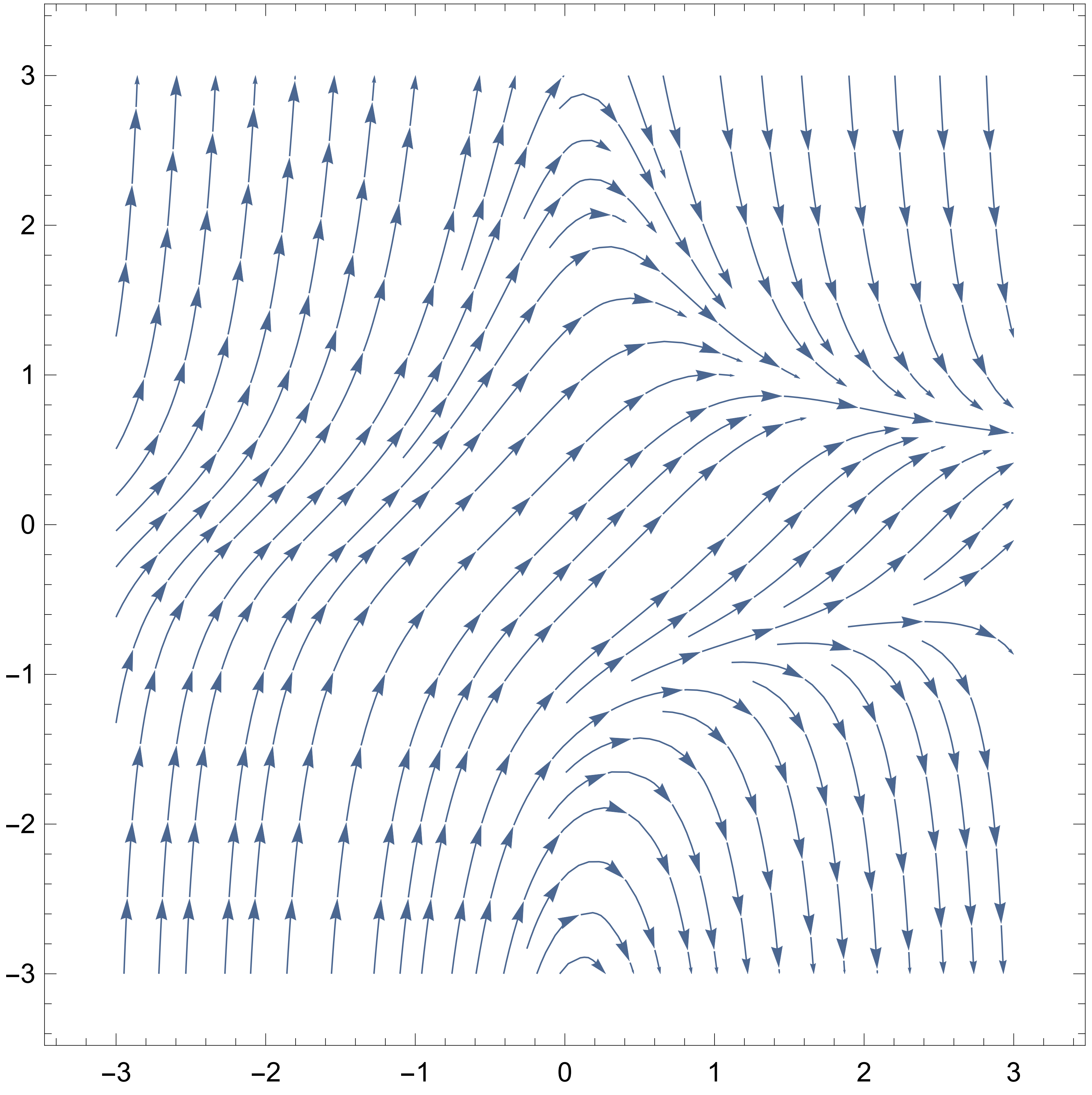
Phase portrait: regular solutions

By plotting the tangent (or vector) field, we may ascertain the features of the solution.

Idea: Take a point M (x, y) of the plane. The path that passes through this point has a slope of $m = f(x, y)$ and we can draw a short segment of slope m issuing from M. By covering the plane with these segments, we can guess the shape of the solution by connection up the segments.

Today, this is quite easy to do as Mathematica and Matlab (and others) have built-in functions.

Phase portrait: regular solutions



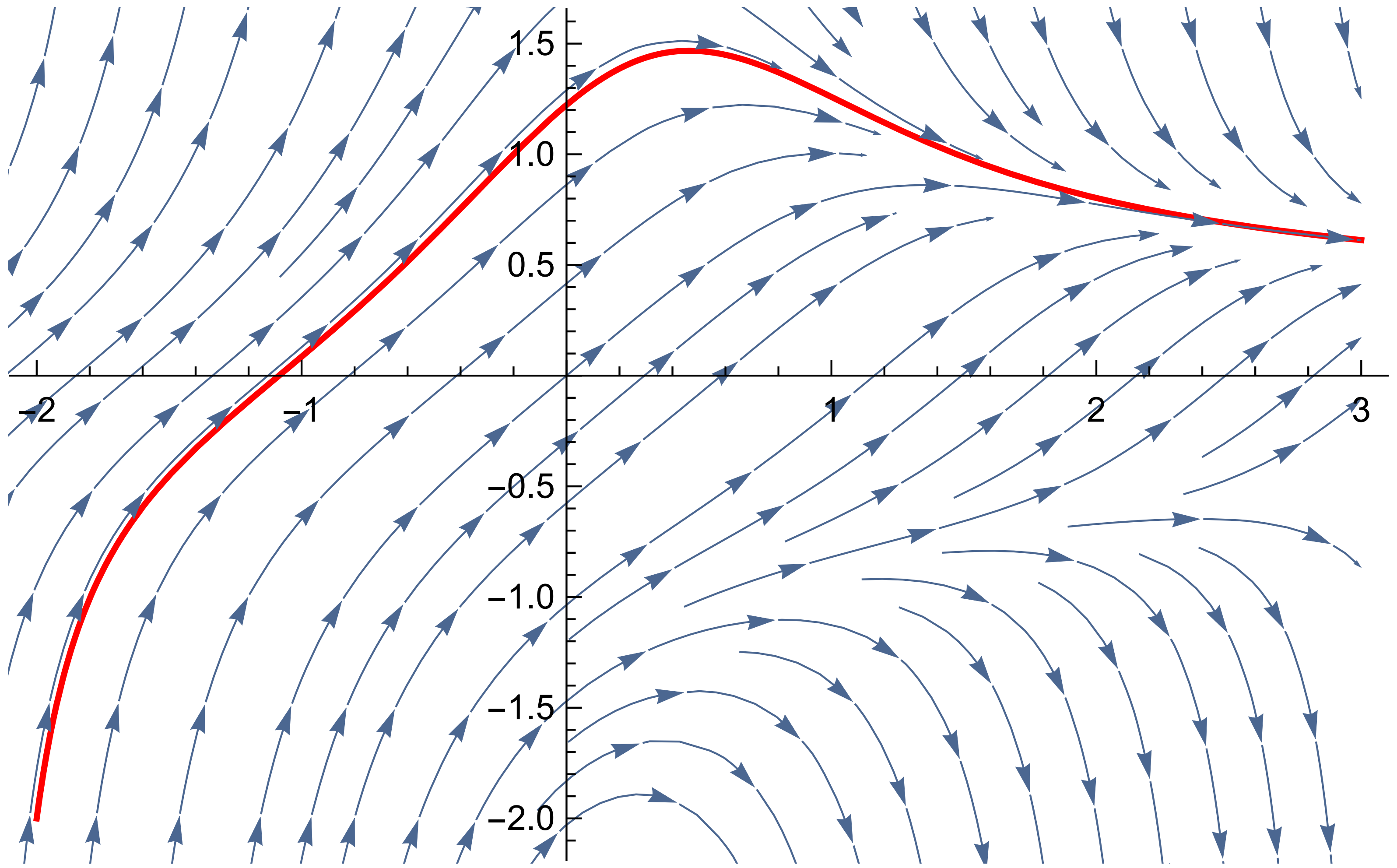
Consider the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - xy^2$$

subject to $y(-2) = -2$. With Mathematica, we can find the main features of the solution by plotting the associated tangent field using *StreamPlot*. With Matlab, the vector field is plotted using *quiver*. For a first-order ODE in the form $y' = N/D$, the syntax is the following:

StreamPlot$[\{D, N\}, \{x, x_{min}, x_{min}\}, \{y, y_{max}, y_{max}\}]$.

Phase portrait: Mathematica



portrait.nb * - Wolfram Mathematica 10.3

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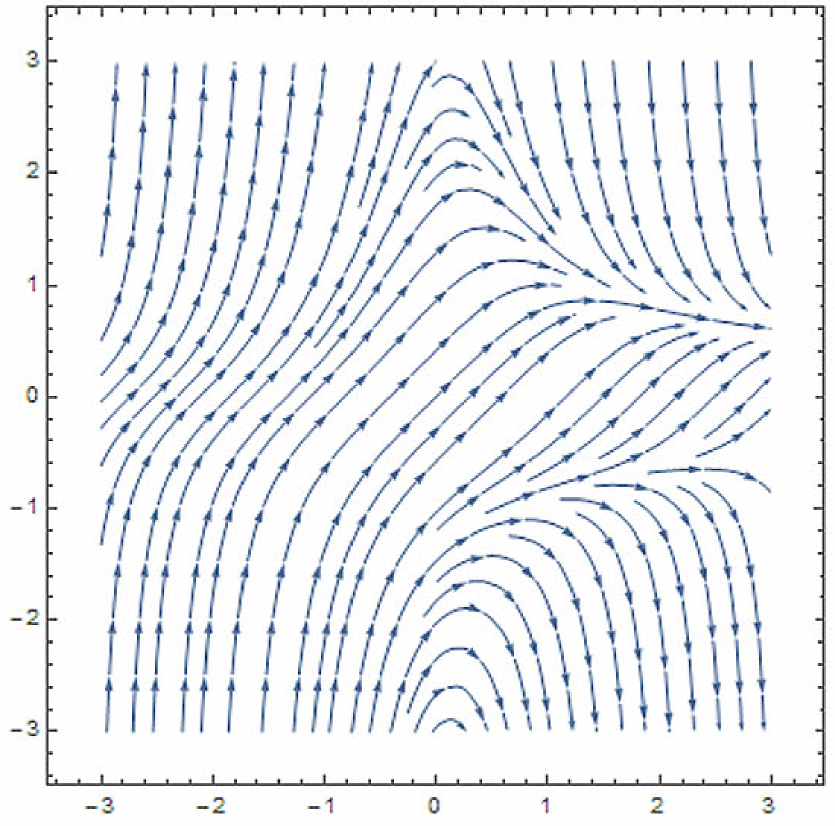
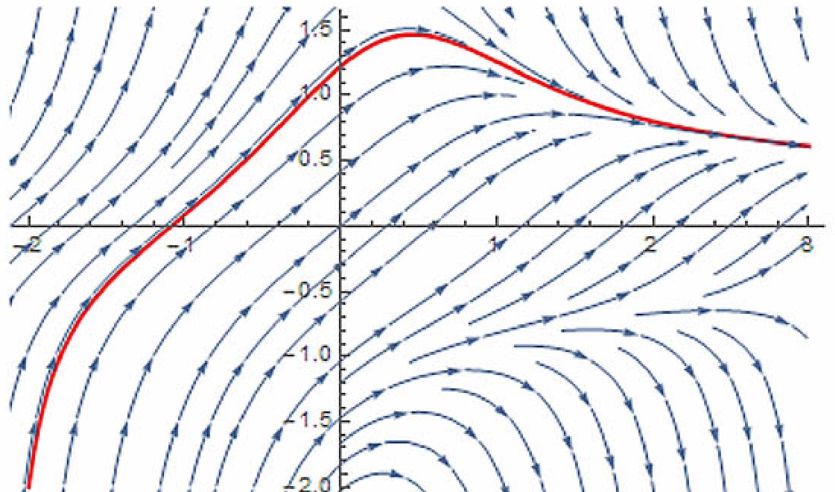
```
In[10]:= fig = StreamPlot[{1, 1 - x y^2}, {x, -3, 3}, {y, -3, 3}]
```

Out[10]=

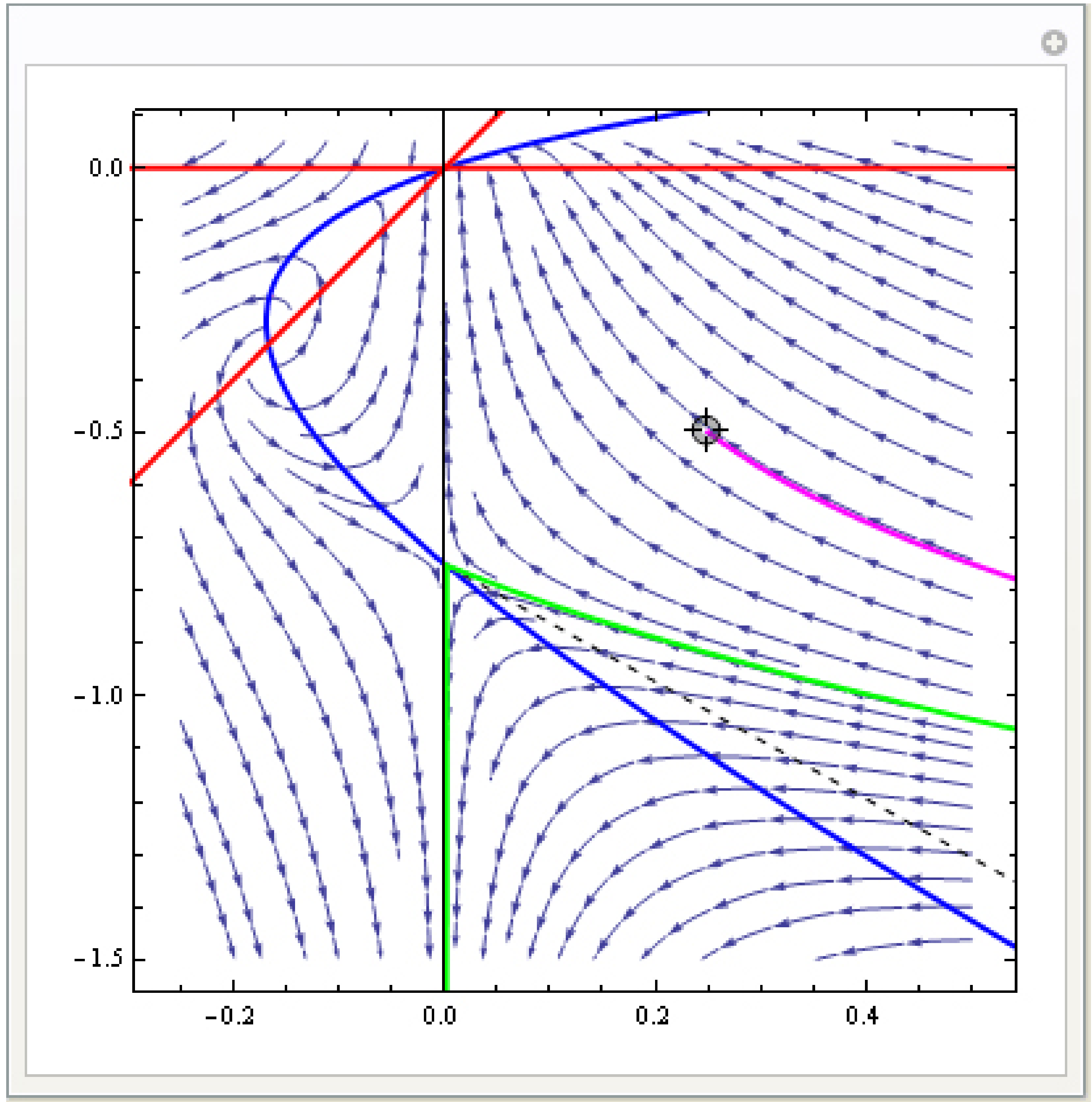


```
SetDirectory["d:"]
Export["portrait.pdf", fig, "PDF"]

Out[12]= D:\

Out[13]= portrait.pdf

In[7]:= sol = y[x] /. DSolve[{y'[x] == 1 - x y[x]^2, y[-2] == -2}, y[x], x][[1]]

Out[7]=  -AiryAi[x] AiryBi[-2] + AiryAi[-2] AiryBi[x] + 2 AiryAiPrime[-2] AiryBi[x] - 2 AiryAi[x] AiryBiPrime[-2]
        ---------------------------------------------------------------------------------------------------------------------------------------
        -AiryAiPrime[x] AiryBi[-2] - 2 AiryAiPrime[x] AiryBiPrime[-2] + AiryAi[-2] AiryBiPrime[x] + 2 AiryAiPrime[-2] AiryBiPrime[x]

In[14]:= fig2 = Show[{Plot[sol, {x, -2, 3}, PlotStyle → Red], fig}]
```

Out[14]=



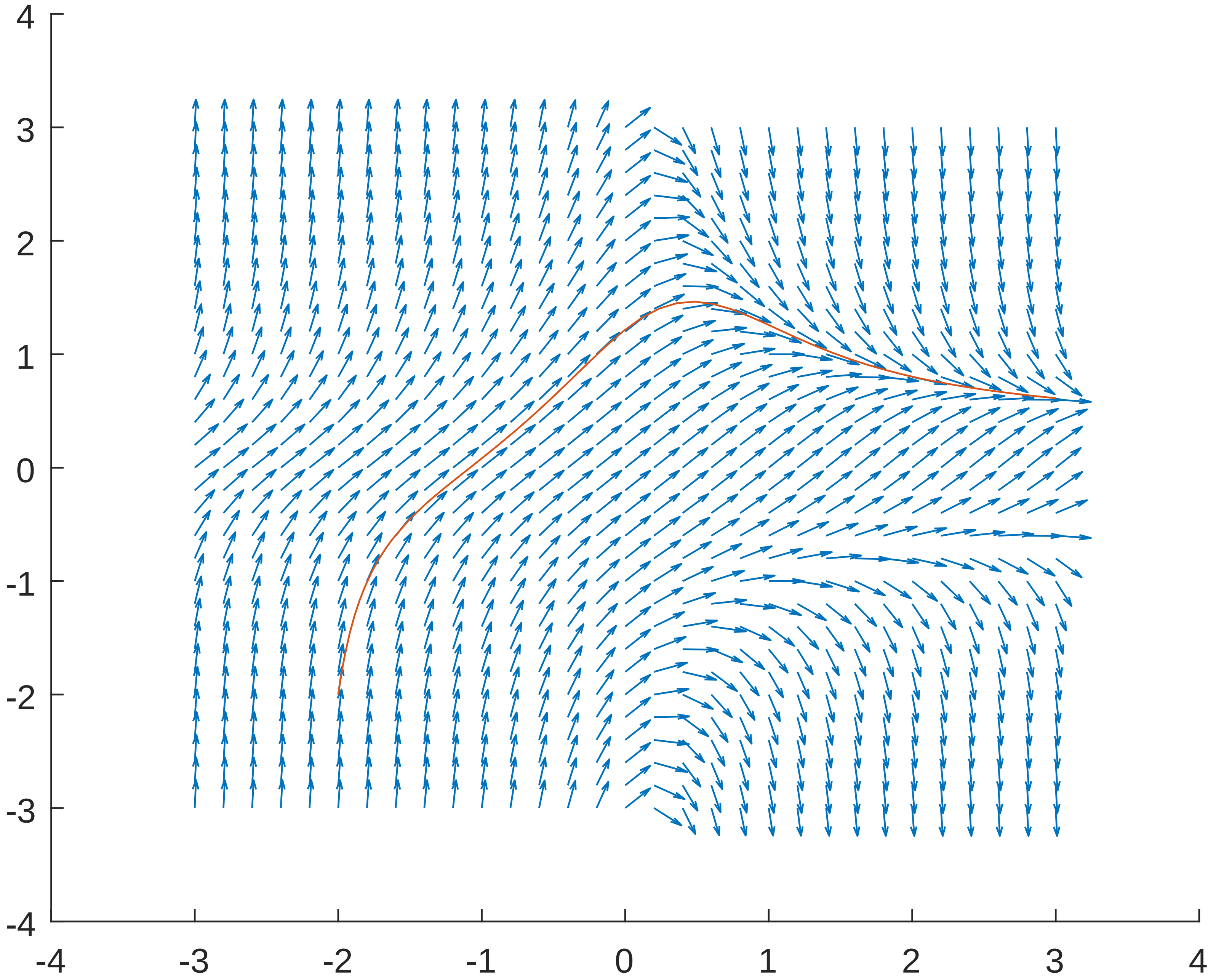
Phase portrait: Mathematica




```
α = 1 / 2
p0 = - (α + 1) / 2
Nu[p_] := p (α + 1 + 2 p) / 2 / (α - p)

Nume[p_, z_] := 2 α z - (1 + α) p - 2 p^2 - 2 p z
Deno[p_, z_] := 2 z (p - 2 z)

solveEqDif3[f_, xy0_, tmax_] :=
 Module[{sol, tp, ti = xy0[[1]]},
  sol = NDSolve[{y'[x] == Nume[y[x], x] / Deno[y[x], x], y[ti] == xy0[[2]]}, y, {x, ti, tmax}];
  tp = sol[[1, 1, 2, 1, 1, 2]];
  Plot[Evaluate[y[x] /. sol], {x, ti, tp}, PlotStyle → {Magenta, Thick}]
 ]

solveEqDifSepBis[f_, tmax_] :=
 Module[{sol, tp, ti = 0, eps = 10^-3},
  sol = NDSolve[{y'[x] == Nume[y[x], x] / Deno[y[x], x], y[eps] == - (α + 1) / 2 - (3 α + 1) / (α + 1) eps}, y, {x, ti, tmax}];
  tp = sol[[1, 1, 2, 1, 1, 2]];
  Plot[Evaluate[y[x] /. sol], {x, ti, tp}, PlotStyle → {Green, Thick}]
 ]
Manipulate[
 Quiet@Show[
   StreamPlot[{2 z (p - 2 z), (2 α z - (1 + α) p - 2 p^2 - 2 p z)}, {z, -0.25, 0.5}, {p, -1.5, 0.05}, ColorFunction → None],
   ParametricPlot[{Nu[p], p}, {p, 1, -4}, PlotStyle → {Thick, Blue}],
   Plot[2 z, {z, -3, 3}, PlotStyle → {Thick, Red}],
   Plot[0, {z, -3, 3}, PlotStyle → {Thick, Red}],
   Plot[- (α + 1) / 2 - (3 α + 1) / (α + 1) 2 / 3 z, {z, 0, 3}, PlotStyle → {Dashed, Black}],
   solveEqDif3[f, pp, 5], solveEqDifSepBis[f, 8], Axes → True
  ],
 {{pp, {0.25, -0.5}}, Locator}]
```

See the website. Notebook for the homework

Phase portrait: Matlab



Editor - D:\portrait.m

portrait.m

```
[x,y] = meshgrid(-3:0.2:3,-3:0.2:3);
n=size(x);
u = ones(n);
v = 1-x.*y.^2;

for i = 1:numel(x)
rac = 2*sqrt(u(i)^2 + v(i)^2);
u(i) = u(i)/rac;
v(i) = v(i)/rac;
end

xspan = [-2 3];
x0 = -2;
[xx,yy] = ode45(@(x,y) 1-x*y^2, xspan, x0);

figure
hold on
quiver(x,y,u,v)
plot(xx,yy,'-')
```

Exercise 1

1. Consider the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y(y^2 - x)}{x^2}.$$

Determine the critical points. Plot the phase portrait.

Exercise 2

2. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \sin y = 0,$$

subject to $y(0) = 1$ and $y'(0) = 1$. What are the features of the solution to this ODE (hint: transform the problem into a first-order differential equation and plot the tangent field). Try to solve the equation analytically (e.g. using DSolve in Mathematica).

Phase portrait: Singular points

The simplest case is encountered when, near the singularity, it is possible to linearize the ODE. We can then write:

$$f(x,\ y) = ax + by + o(x,\ y) \text{ and } g(x,\ y) = cx + dy + o(x,\ y).$$

Let us assume that $ad - bc \neq 0$ and these coefficients are not all zero.

There are two critical curves in the vicinity of the singularity:

- $y = -ax/b$ where the curves admit a horizontal tangent;
- $y = -cx/d$ where the curves admit a vertical tangent.

Phase portrait: Linearization

Introducing the dummy variable t, we can transform the linearized ODE into the matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = \boldsymbol{M}\cdot\boldsymbol{u}, \text{ with } \boldsymbol{M} = \begin{bmatrix} c & d \\ a & b \end{bmatrix},$$

where $\boldsymbol{u} = (x, y)$. We seek a solution in the form $\boldsymbol{v} = \boldsymbol{v}_0 \exp(\lambda t)$, with $\boldsymbol{v}_0$ the initial-condition vector (at $t = 0$). λ must be an eigenvalue of the matrix $\boldsymbol{M}$ and $\boldsymbol{v}_0$ an associate eigenvector; λ is solution to the second order equation $\lambda^2 - 2h\lambda + k = 0$, with $2h = b + c$ and $k = \det \boldsymbol{M} = -ad + bc$, i.e.:

$$\lambda = h \pm \sqrt{h^2 - k}.$$

The principal directions are: $2a/(b - c \mp \sqrt{(b-c)^2 + 4ad})$ (they are given by the eigenvectors).

Phase portrait: Node

Depending on the value taken by λ, different behaviours arise.

When $\Delta = h^2 - k > 0$ and $k > 0$, the two eigenvalues are real and of the same sign. Assume that $h > 0$, then the two eigenvalues are positive, which means that either both solutions to the ODE tend to 0 as $t \to -\infty$ (resp. when $h < 0$, the solutions tend to 0 as $t \to +\infty$). Hence, if every initial condition lies on one of the principal axes, each solution tends towards the origin point, the solution is part of a straight line with slope equal to one of the principal directions.

What happens if the initial condition does not lie on one of the principal directions?

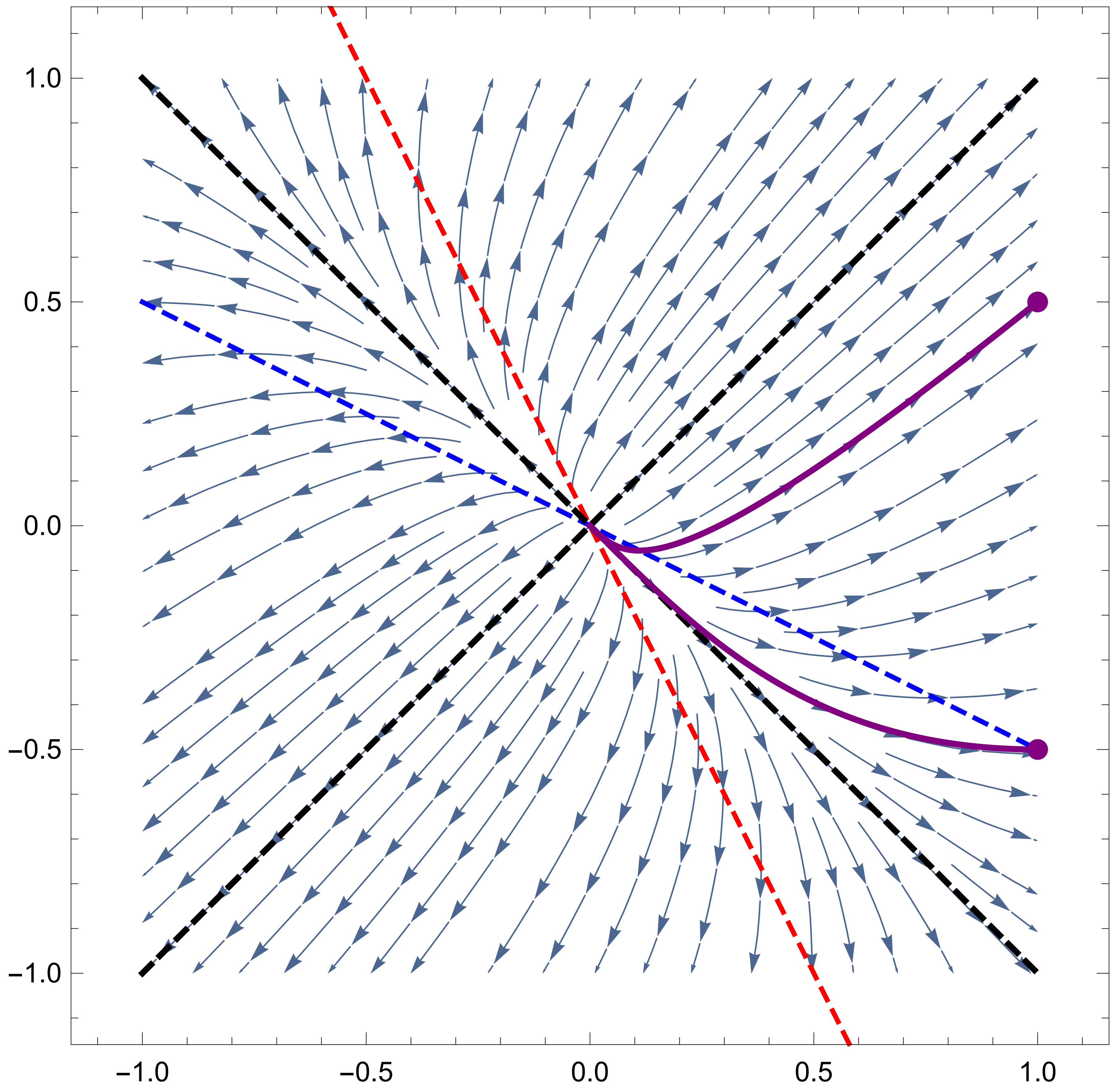
Phase portrait: Node

Let us assume that an integral curve tends towards the origin point. The limit of $\mathrm{d}y/\mathrm{d}x$ at 0 in the equation is not defined. Upon application of L'Hospital's rule, the slope of the solution at point O must satisfy:

$$m = \frac{a + bm}{c + dm},$$

i.e., $m = b - c \pm \sqrt{(b-c)^2 + 4ad}$ and m coincides with one of the main directions. Given the sign of $\mathrm{d}y/\mathrm{d}x$ around the origin point, only one of these solutions is possible: the curves reach the origin point, following an asymptotic curve of equation $y = mx$. This singularity is called a *node*.

Phase portrait: Node



When solving the equation:

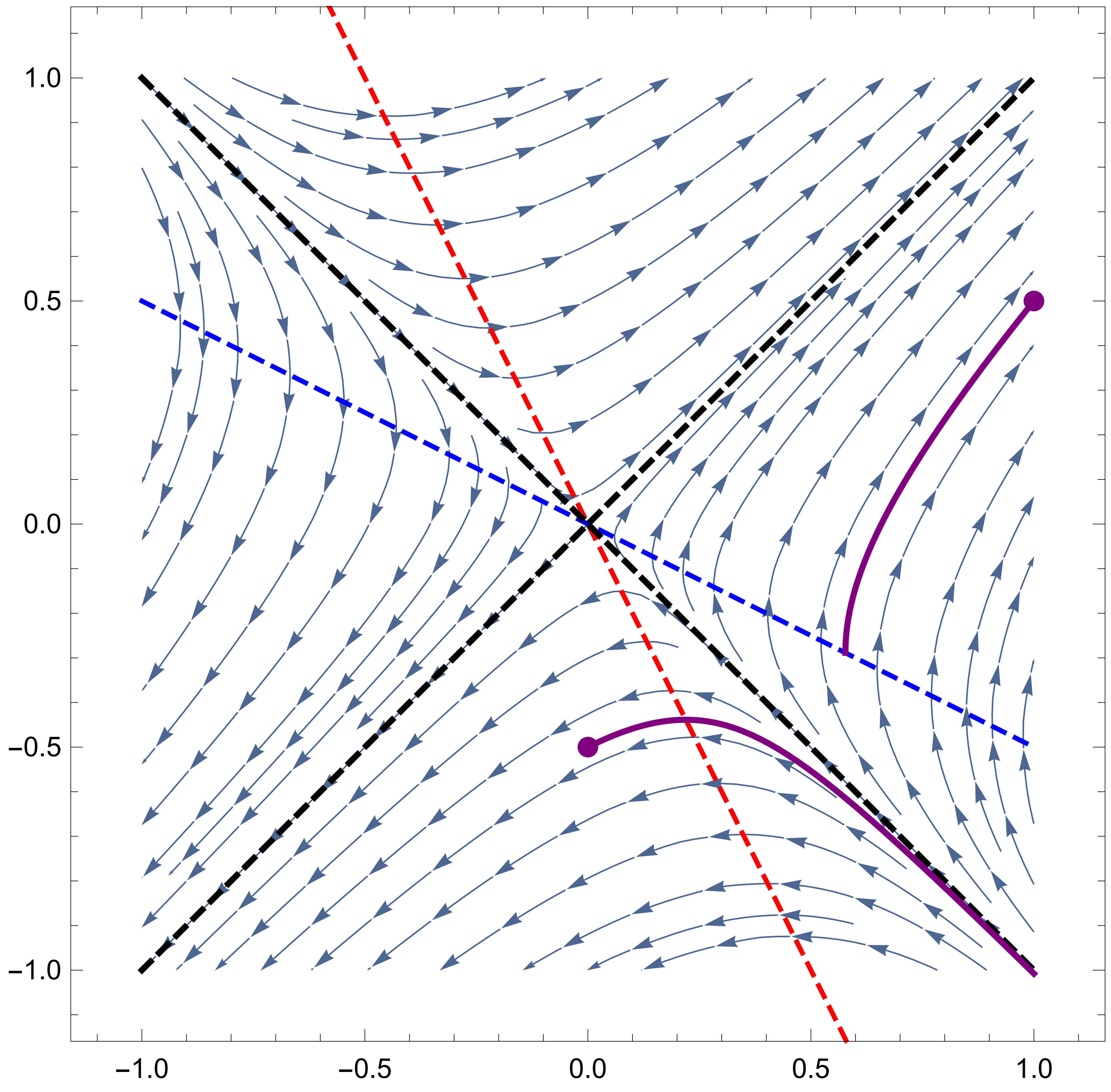
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + 2y}{2x + y},$$

we find that there are two eigenvalues 3 and 1 associated with principal directions 1 and -1 respectively. It is a node. Solid purple lines are solutions to the differential equation. Blue and red lines represent the singular curves while the black dashed lines stand for the principal directions.

Phase portrait: Saddle point

If $\Delta > 0$ and $k < 0$, the two eigenvalues are real and of opposite sign. The two solutions of the linearized ODE behave differently when $t \to \infty$: one tends towards the singular point, whereas the other tends to infinity. There are always two curves that pass through the singular point and that coincide with the principal directions. If now the initial point (i.e., the initial condition of the differential equation) does not lie on one of the principal directions, then it is not possible to find an integral curve going from that point to the singular point because of the sign of $\mathrm{d}y/\mathrm{d}x$ in the close vicinity of the singular point. The paths diverge when approaching the singular point. We refer to this point as a *saddle point*.

Phase portrait: Saddle point



When solving the equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x + y}{x + 2y},$$

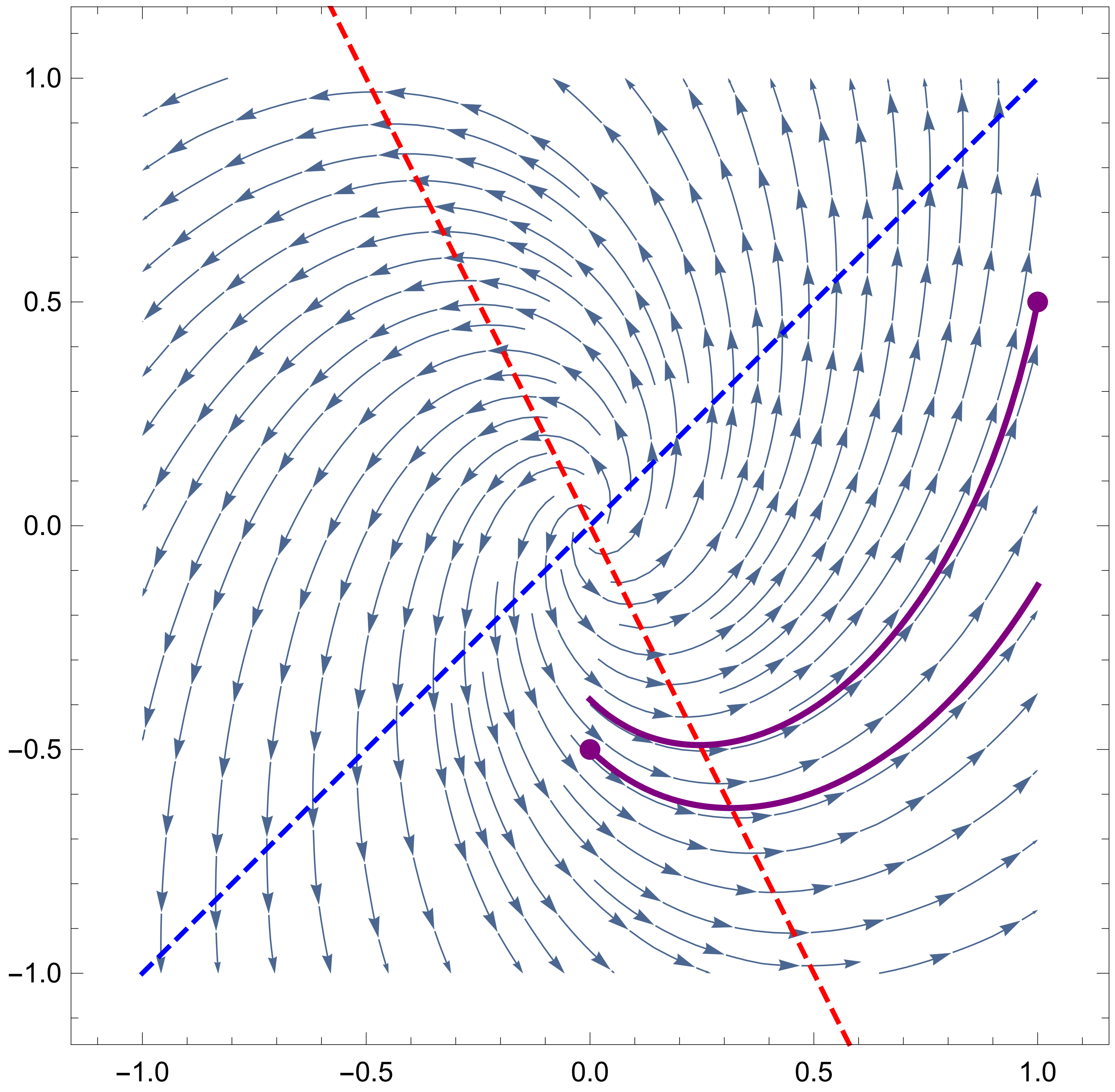
we find that there are two eigenvalues 3 and -1 associated with principal directions 1 and -1 respectively. It is a saddle point.

Phase portrait: Focal point

If $\Delta = 0$, the singular point is a node.

If $\Delta < 0$, both eigenvalues are imaginary. The curves coiled like a spiral around the singular point. This point is called *focal point*.

Phase portrait: Focal point

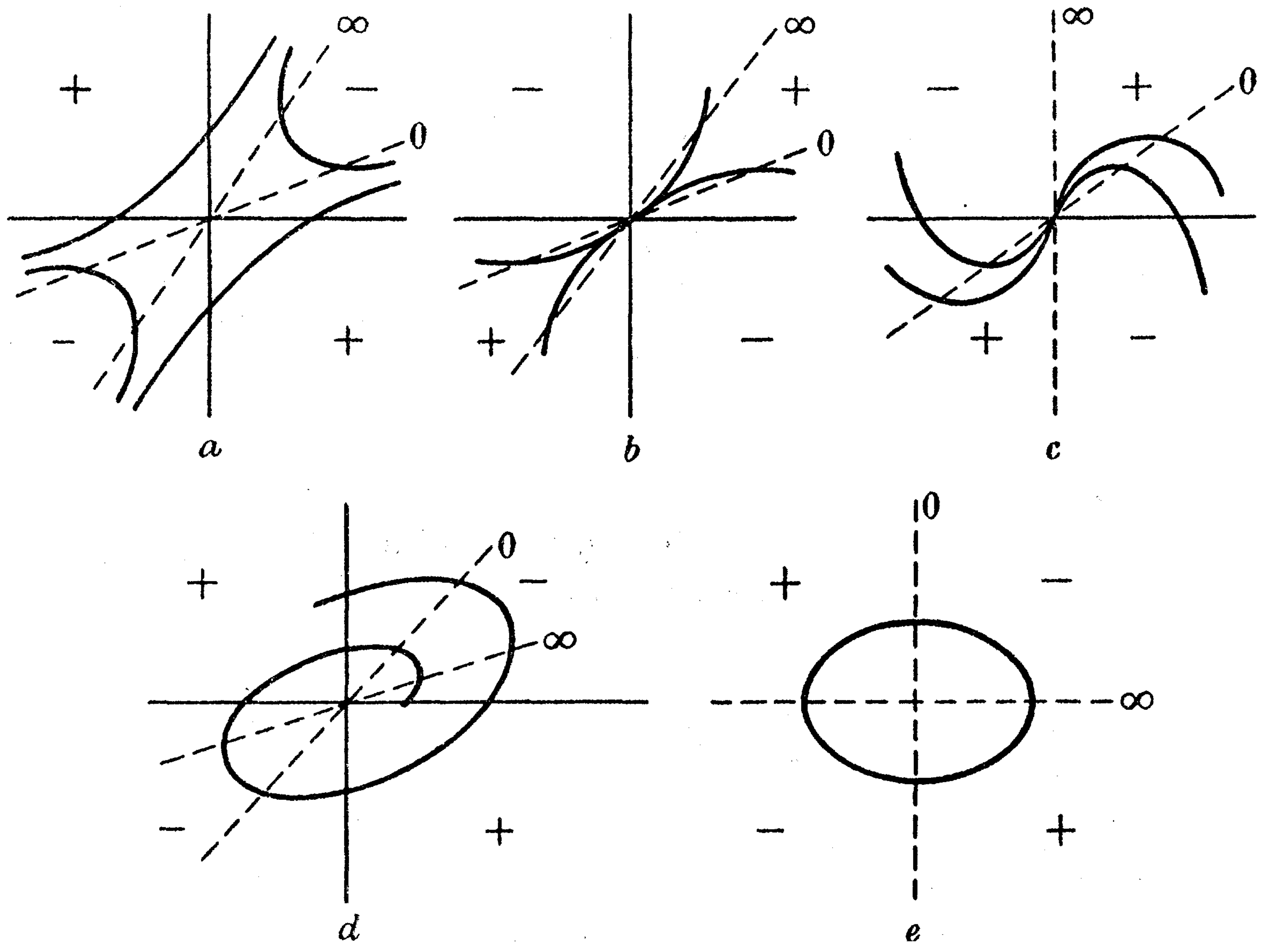


When solving the equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x + y}{x - y},$$

we find that there are two complex eigenvalues $(3 \pm \imath\sqrt{3})/2$ (and eigenvectors are also complex). It is a *focal point*.

Phase portrait: Typology of singular points

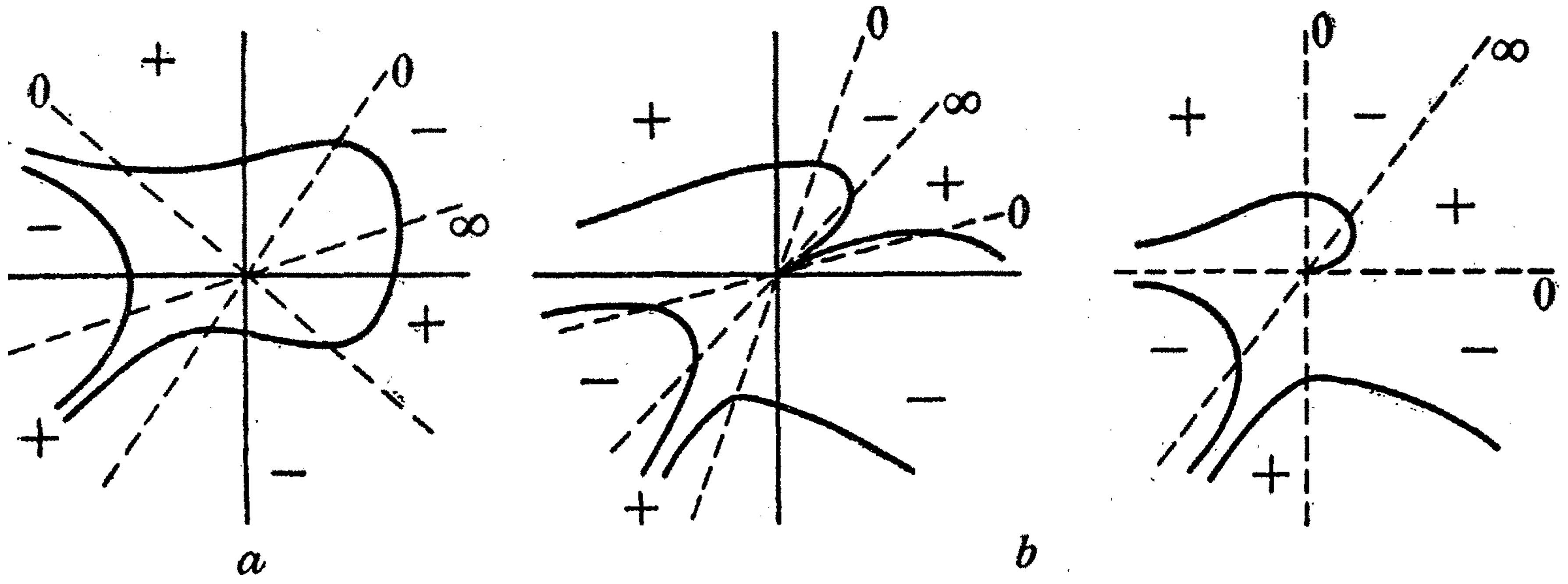


Typology of singular points where there are two critical curves (Jones, C.W., On reducible non-linear differential equations occurring in mechanics,, Proceedings of the Royal Society of London A, 217, 327-343, 1953.)

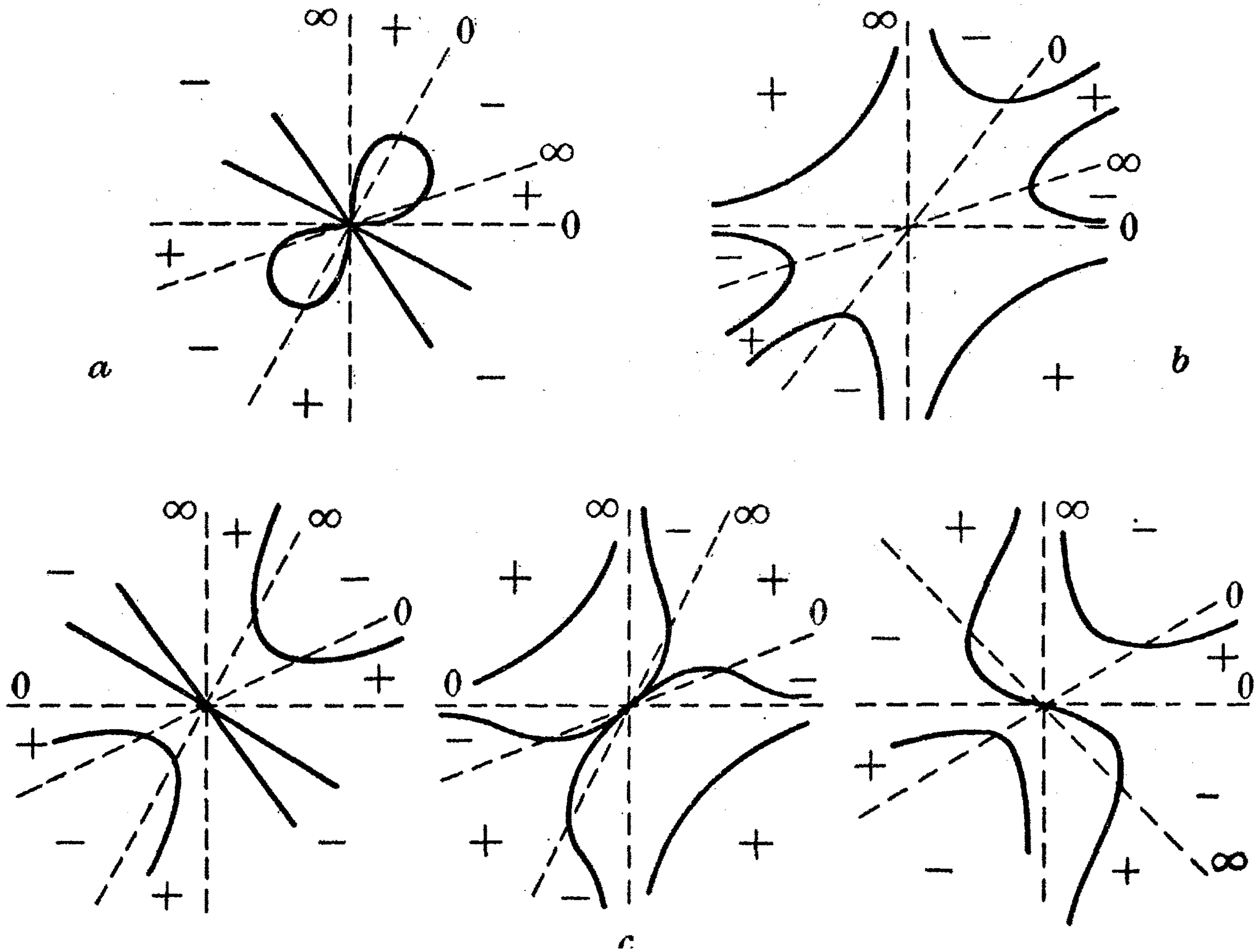
Generalization. Three possible types of behaviour:

- *node* where the integral curves are directed towards the singular point, usually following an asymptotic curve that can be deduced from the differential equation
- *saddle* where the integral curves diverge as they approach the singular point, except one which is able to cross it
- *focal point* where the curves wrap up like spirals or make loop around the singular point

Phase portrait: Typology of singular points



Typology of singular points where there are three critical curves



Typology of singular points where there are four critical curves

Phase portrait: Separatrix

When the singularity is a node, there is an asymptotic curve to which any curve passing through the singularity tends. Similarly, when the singular point is a saddle, there is a (single) curve solution that arrives at the singular point. The exceptional curve is called *separatrix* because it separates two regions of space, each characterized by a specific behaviour near the singular point. We can use several methods to determine the equation of this curve.

Analytical approach. A separatrix is a curve, which is solution to the ODE and invariant under any Lie group admitted by the same equation. An implicit curve equation $\phi(x,\ y) = 0$ is invariant under a Lie group $\Gamma = \xi\partial_x + \eta\partial_y$ if $\Gamma\phi = 0$.

Phase portrait: Separatrix

Using the condition

$$\Gamma\phi = \xi\partial_x\phi + \eta\partial_y\phi = 0,$$

we deduce that ϕ is also solution to the following first-order differential equation

$$y' = \frac{\eta(x,\ y)}{\xi(x,\ y)}.$$

The equation of the separatrix is then obtained by substituting y' by η/ξ into the ODE:

$$\frac{\eta(x,\ y)}{\xi(x,\ y)} = \frac{f(x,\ y)}{g(x,\ y)}.$$

To find the equation of the separatrix, we must then find all groups that leave the ODE invariant. This is beyond the scope of the course and we mention this technique just for completeness.

Phase portrait: Separatrix

Example. Consider the differential equation

$$y' = \frac{y(x - y^2)}{x^2},$$

which is invariant under transformation $x_1 = \lambda x$ and $y_1 = \sqrt{\lambda} y$, whose infinitesimal generator is $\Lambda = 2x\partial_x + y\partial_y$. We then deduce $\xi = 2x$ and $\eta = y$. The equation of the separatrix is

$$\frac{y}{2x} = \frac{y(x - y^2)}{x^2},$$

or, equivalently,

$$y^2 = \frac{x}{2}.$$

Phase portrait: Separatrix

Numerical approach. Using L'Hôpital's rule, we can obtain the sepatrix equation:

$$F(x) = f(x,\ y(x)) \quad \text{and} \quad G(x) = g(x,\ y(x)).$$

By making a first-order expansion around a singular point $\boldsymbol{x}_s$, we get:

$$\dot{y}_s + x\ddot{y}_s + \cdots = \frac{x\dot{F}_s + \dfrac{x^2}{2}\ddot{F}_s + \cdots}{x\dot{G}_s + \frac{x^2}{2}\ddot{G}_s + \cdots} = \frac{\dot{F}_s + \dfrac{x}{2}\ddot{F}_s + \cdots}{\dot{G}_s + \frac{x}{2}\ddot{G}_s + \cdots},$$

with $\dot{y}_s = \dot{y}(\boldsymbol{x}_s)$ et $\ddot{y}_s = \ddot{y}(\boldsymbol{x}_s)$.

See Ancey *et al.*, Existence and features of similarity solutions for supercritical non-Boussinesq gravity currents, Physica D, 226, 32-54, 2007.

Phase portrait: Separatrix

Computing $\dot{F}_s$ involves compound derivatives:

$$\dot{F} = \frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y},$$

$$\ddot{F} = \frac{\partial \dot{F}}{\partial x} + \dot{y}\frac{\partial \dot{F}}{\partial y} + \ddot{y}\frac{\partial \dot{F}}{\partial \dot{y}}.$$

We do the same with G. We wish to compute the series expansion of the asymptotic curve at the singular point, that is, an equation of the form $y = y_s + m(x - x_s) + p(x - x_s)^2/2$, with $m = \dot{y}_s = \dot{y}(x_s)$ and $p = \ddot{y}_s = \ddot{y}(x_s)$. To order 0, we must solve the second-order equation:

$$m = \frac{f_x + m\ f_y}{g_x + m\ g_y},$$

to find m.

Phase portrait: Separatrix

Once m is known, we can infer p, which is solution to the following equation:

$$\ddot{F}_s = m\ddot{G}_s + 2p\dot{G}_s,$$

from which we deduce p.

The higher order terms are determined iteratively using the same technique.

Example. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + 3xy + 3(1-y)y}{3x(2x+3y)}.$$

Phase portrait: Separatrix

We would like to determine how the integral curves behave close to the origin point (which is singular). We find:

$$\dot{F}(0) = 1 + 3m \quad \text{and} \quad \dot{G}(0) = 0,$$

and the solution is $m = -1/3$. To the following order, we get

$$\ddot{F}(0) = -\frac{8}{3}p \quad \text{and} \quad \ddot{G}(0) = 6.$$

And thus we find $p = 2/9$. The sepatrix has the following asymptotic equation to order 2

$$y = -\frac{1}{3}x\left(1 - \frac{1}{3}x + \dots\right).$$

Exercise 3

Exercise 3. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + 3xy + 3(1-y)y}{3x(2x+3y)}.$$

Determine how the integral curves behave close to the origin point. Plot the sepatrix.

Phase portrait: Singular points at infinity

When $f(x,\ y) \to \infty$ and $g(x,\ y) \to \infty$ when $x \to \infty$ and $y \to \infty$, the behaviour of $\mathrm{d}y/\mathrm{d}x$ is indefinite there. One way to find the proper limit is to use the dominant-balance technique, i.e. $y \ll x$, $y \sim x$, or $y \gg x$, then integrate the resulting differential equation to check whether the assumption is consistent *a posteriori* or not.

Singular points expelled to infinity may actually represent a single point; a change of variable can usually show that. For example, with the following variable change

$$x_1 = \frac{x}{x^2 + y^2} \text{ et } y_1 = -\frac{y}{x^2 + y^2},$$

then by analyzing behavior at (0, 0) in the $(x_1,\ y_1)$ plane, we can determine the behaviour of a singular point at infinity (note that it is equivalent to making the change $z_1 = 1/z$ with $z = x + \imath y$).

Exercise 4

4. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y(x-2y)}{(1-3x)y - x - x^2},$$

Investigate its behaviour in the limit $|x| \to \infty$ and $|y| \to \infty$. Determine the asymptotic behaviour(s) at infinity.

Phase portrait: Singular points with $m = 0$ or ∞

We can face differential equations with singularities satisfying $m = 0$ or $m = \pm\infty$. The tangent is then horizontal or vertical, respectively. In this case, the behaviour is deduced by approximation and integration of the solution (argument like the one used in the *dominant balance* technique).

Example. Let us consider the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8 - 3x}{x(4 - x) - 2} y,$$

for which we note that the denominator vanishes at A_- $(2 - \sqrt{2}$, 0) and A_+ $(2 + \sqrt{2}$, 0), which are two singular points (nodes). The numerator vanishes at A_0 $(8/$ 3, 0), which corresponds to an extremum in the integral curve.

Phase portrait: Singular points with $m = 0$ or ∞

The solution has the following behaviour:

		A_-		A_0		A_+	
numerator	$+$	$\vert$	$+$	$\vert$	$-$	$\vert$	$-$
denominator	$-$	$\vert$	$+$	$\vert$	$+$	$\vert$	$-$
function	$-$	$\vert$	$+$	$\vert$	$-$	$\vert$	$+$

Phase portrait: Singular points with $m = 0$ or ∞

The behaviour around the nodes is then given by:

- for A_-, we get:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \approx n \frac{y}{x - x_{A-}},$$

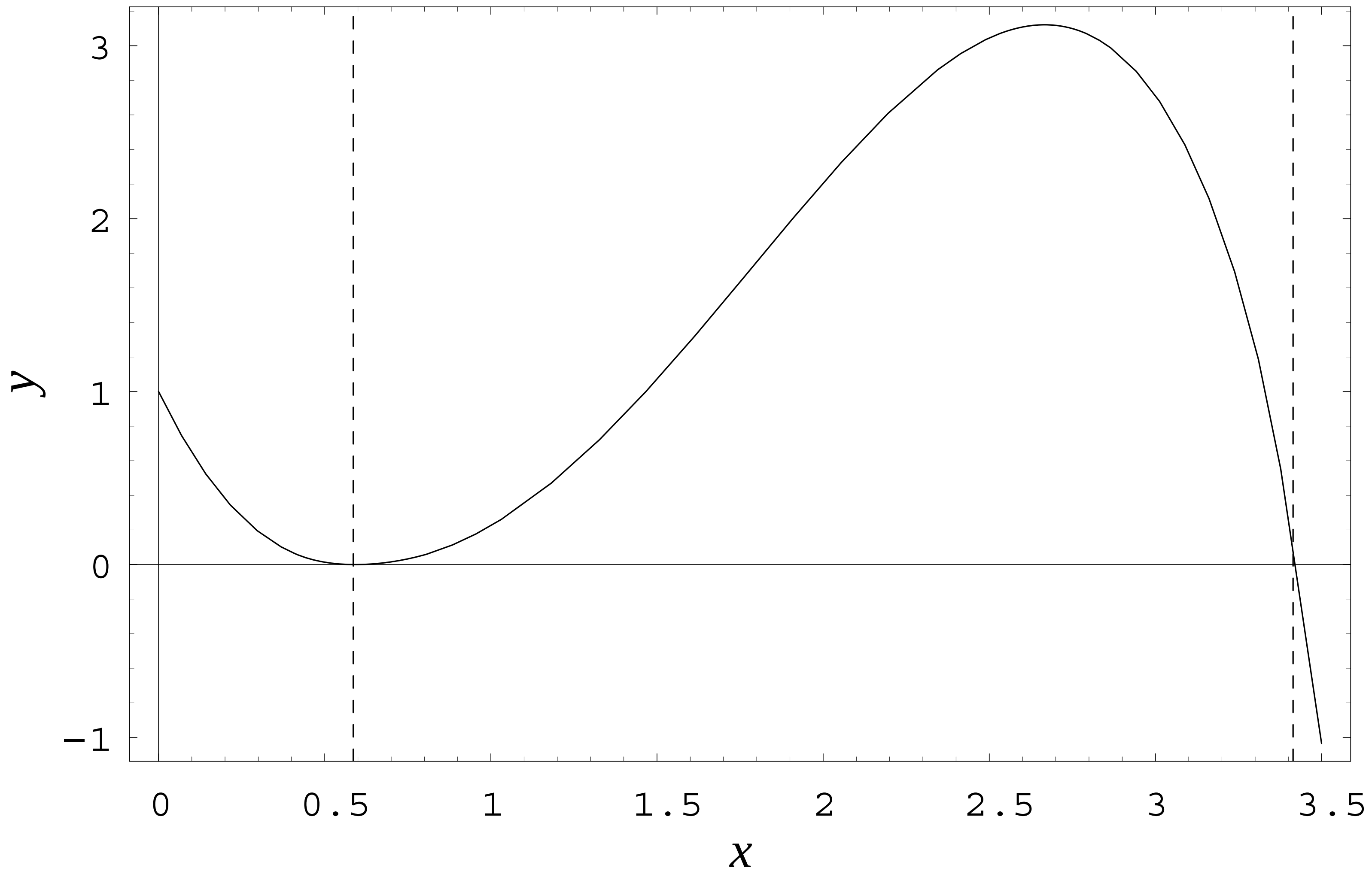
  with $x_{A-} = 2 - \sqrt{2}$ and $n = (8 - 3x_{A-})/(x_{A+} - x_{A-}) = 3/2 + 1/\sqrt{2} \approx 2,20 > 1.$ After integration, we find: $y = c(x - x_{A-})^n$. The curve admits a horizontal tangent.

- for A_+, we have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \approx n' \frac{y}{x - x_{A+}},$$

  with $n' = (3x_{A+} - 8)/(x_{A+} - x_{A-}) = -3/2 + 1/\sqrt{2} \approx 0,79 < 1.$ After integration, we get: $y = c|x_{A+} - x|^{n'}$. The curve admits a vertical tangent.

Phase portrait: Singular points with $m = 0$ or ∞



Note that in this case, there is an analytic solution of the form:

$$y(x) = c|2 - ax + x^2|^{3/2} \exp\left[-\sqrt{2}\,\mathrm{arctanh}\frac{x-2}{\sqrt{2}}\right].$$

The result (numerical integration) is reported in the figure. Note that the vertical tangent at point A_+ is not very apparent because the interval over which the derivative is very large is narrow.

Exercise 5

5. Consider the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6y(2y-x)}{2q^2+6yx+x}.$$

(q a free parameter). Study its behaviour near 0.

Lie's integrating factor

Let us consider a first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

that is invariant to a group Γ. How helpful is this information in determining a solution to this ODE?

Let us write the ODE differently

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \Leftrightarrow \mathrm{d}y - f(x, y)\mathrm{d}x = 0.$$

A solution can be represented parametrically by an equation of the form $\psi(x, y) = c$ with c the parameter. The total derivative is

$$\psi_x \mathrm{d}x + \psi_y \mathrm{d}y = 0.$$

Lie's integrating factor

If we divide it by the arbitrary function $\mu(x, y)$, then we get

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0 \text{ with } M = \frac{\psi_x}{\mu} \text{ and } N = \frac{\psi_y}{\mu}$$

Conversely, if we have this equation, we need to multiply it by μ to transform into a perfect differential. For this reason, we refer to μ as the *integrating factor*. At the same time $\Gamma\psi = 1$, and if ξ and η are the infinitesimal coefficients, then we also have

$$\xi\psi_x + \eta\psi_y = 1.$$

Since $\psi_x = \mu M$ and $\psi_y = \mu N$, we deduce that

$$\boxed{\mu = \frac{1}{\xi M + \eta N}.}$$

Exercise 6

6. Consider the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y(y^2 - x)}{x^2}.$$

Show that this ODE is invariant to the stretching group. Deduce the integrating factor and solve the ODE.

Singular curve

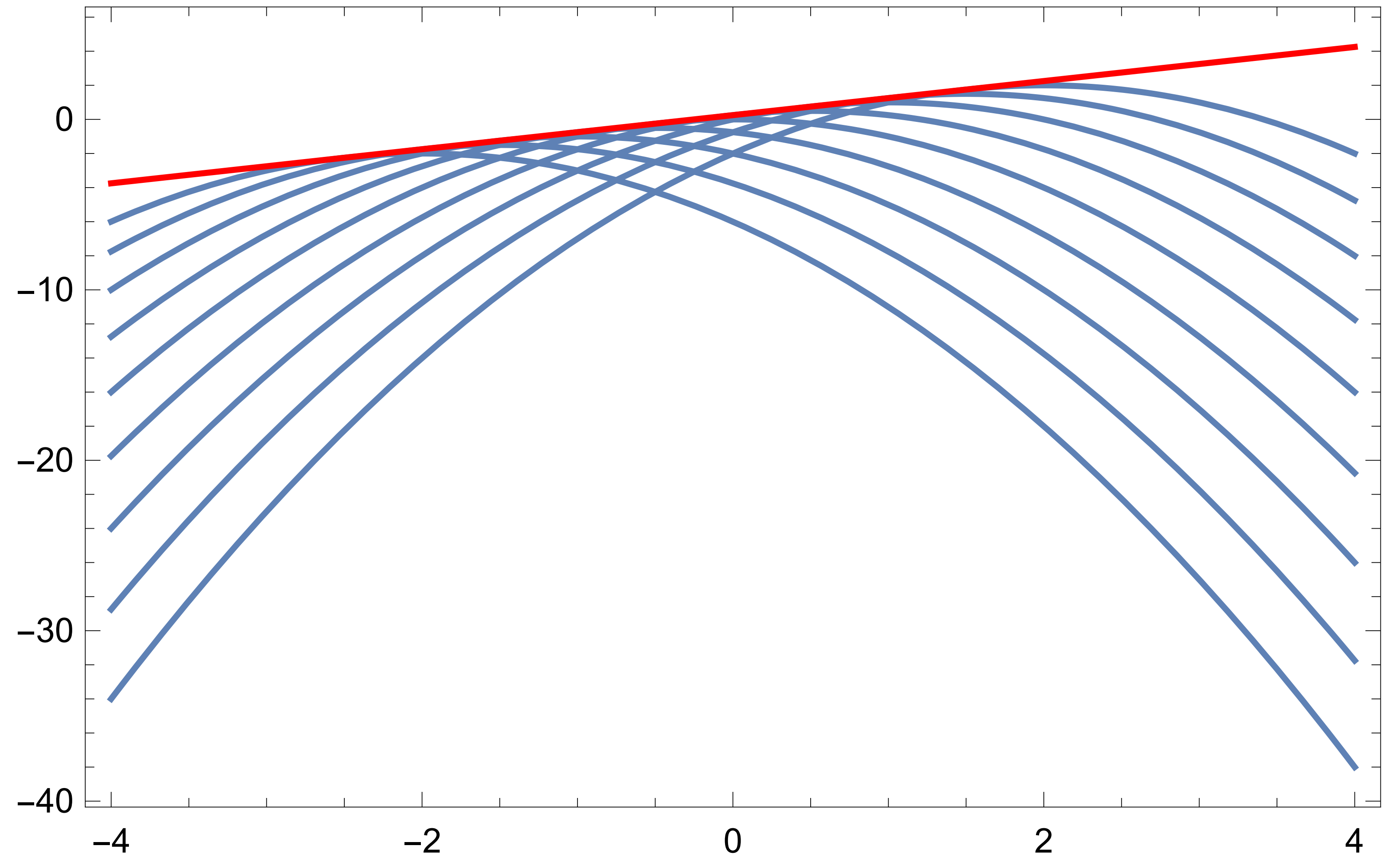
Implicitly we have so far assumed that at regular points (points at which M and N do not vanish simultaneously), there is a single tangent. This is not always the case. For instance, the equation

$$\dot{y}^2 - 2\dot{y} + 4y - 4x = 0$$

has not a uniquely determined tangent at each point of the plane (x, y). The discriminant of the second-order polynomial is $\Delta = 4 - 16(y - x)$. So, when $\Delta > 0$ (i.e. $y < x + 1/4$), there are two solutions, for $\Delta = 0$ (i.e. $y = x + 1/4$), a single solution, and for $\Delta < 0$, none. The solution to this ODE is

$$y = c - (x - c)^2.$$

Singular curve

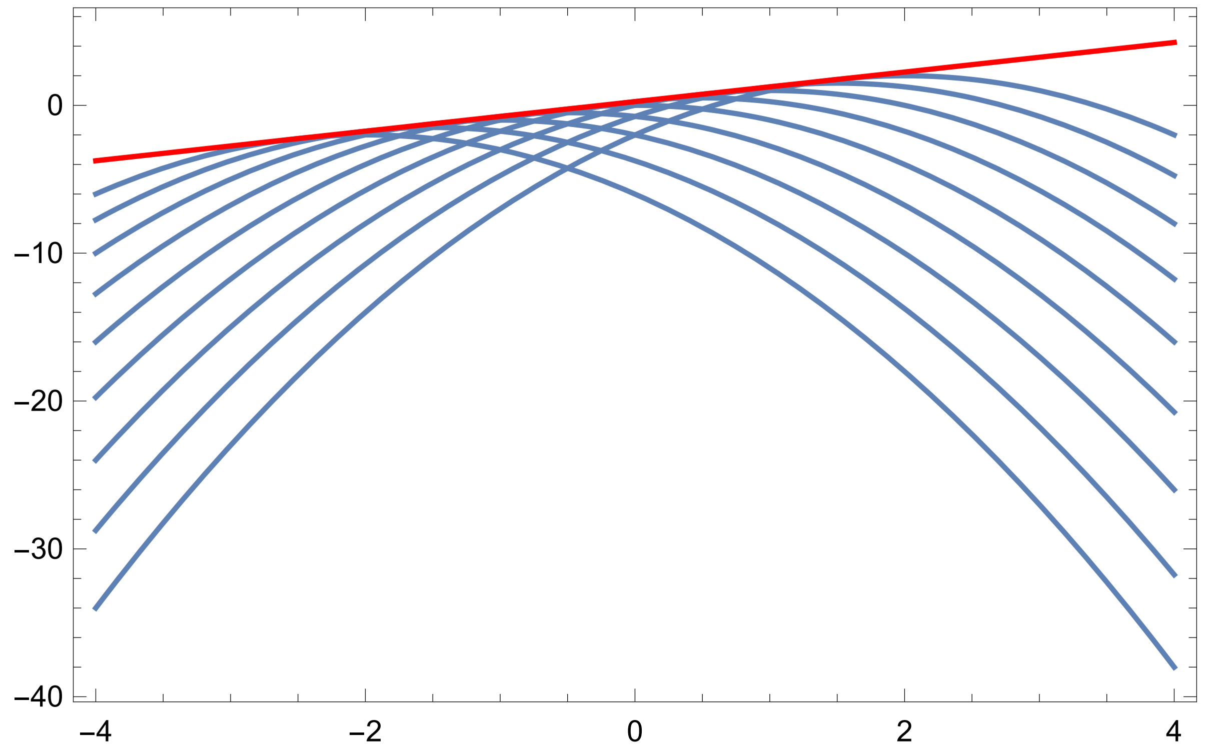


The family of solutions has the *envelope*

$$y = x + \frac{1}{4}$$

An envelope is a curve that is tangent at each of its points to a member of the family. It thus also satisfies the ODE. This is a *singular solution*.

Reminder: envelope equation



When the envelope of the one-parameter family of curves $F(x, y, \lambda) = 0$ exists, its equation is given by the system

$$\begin{cases} F(x, y, \lambda) = 0 \\ \dfrac{\partial F}{\partial \lambda}(x, y, \lambda) = 0 \end{cases}$$

Exercise 7

7. Consider the differential equation (Clairaut's equation):

$$x\dot{y}^2 - y\dot{y} + m = 0.$$

with m a free parameter. Show that this ODE is invariant to the stretching group. Determine the solutions to this equation.

Change of variables

Symmetries can help find a change of variables and solve the ODE. Let us introduce the new variables

$$\hat{x} = F(x, y) \text{ and } \hat{y} = G(x, y)$$

where F and G are to be determined. We consider the image of (x, y) by the group transformation

$$x' = X(x, y; \lambda) \text{ and } y' = Y(x, y; \lambda).$$

The new coordinates of the image are

$$\hat{x}' = F(X(x, y; \lambda), Y(x, y; \lambda))$$

$$\hat{y}' = G(X(x, y; \lambda), Y(x, y; \lambda))$$

Change of variables

Differentiating the coordinates with respect to λ and setting $\lambda = \lambda_0$

$$\hat{\xi} = \xi F_x + \eta F_y = \mathcal{D} F$$

$$\hat{\eta} = \xi G_x + \eta G_y = \mathcal{D} G$$

where $\mathcal{D}$ is the short-hand notation for $\xi \partial_x + \eta \partial_y$.

Let us assume now that F and G are chosen so that $\hat{\xi} = 0$ and $\hat{\eta} = 1$. Then the coefficient of the extended group is $\eta_1 = 0$, and the invariant of the once-extended group is a function u such that

$$\xi \partial_{\hat{x}} u + \eta \partial_{\hat{y}} u + \eta_1 \partial_{\widehat{\dot{y}}} u = 0 \Rightarrow \partial_{\widehat{\dot{y}}} u = 0$$

which means that $u(\hat{x}, \widehat{\dot{y}}) = 0$ or $\widehat{\dot{y}} = A(\hat{x})$ where A is an arbitrary function. So the transformed ODE is *separable* (the variables are separate).

Change of variables

Theorem 1. If we can solve

$$\xi F_x + \eta F_y = 0$$
$$\xi G_x + \eta G_y = 1$$

then we can make a change of variables that makes the original ODE separable.

Example. Let us consider again

$$\dot{y}^2 - 2\dot{y} + 4y - 4x = 0$$

This equation is invariant to the translation group $x' = x + \lambda$ and $y' = y + \lambda$ associated with $\xi = 1$ and $\eta = 1$. What change of variables? We have to solve

$$\xi F_x + \eta F_y = 0$$
$$\xi G_x + \eta G_y = 1$$

Change of variables (example)

We have to solve

$$F_x + F_y = 0 \Leftrightarrow \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{1}$$

$$G_x + G_y = 1 \Leftrightarrow \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{1} = \frac{\mathrm{d}G}{1}$$

So from the first ODE, we deduce that any function of $x - y$ is solution, while $G - y$ is solution to the second ODE. So we pose

$$\hat{x} = y - x \text{ and } \hat{y} = y$$

We thus have $y = \hat{y}$, $x = \hat{y} - \hat{x}$, $\dot{y} = \widehat{\dot{y}}/(\widehat{\dot{y}} - 1)$. The transformed ODE is

$$\widehat{\dot{y}} = 1 \pm (1 - 4\hat{x})^{-1/2} \Rightarrow \hat{y} = \hat{x} \pm \frac{1}{2}(1 - 4\hat{x})^{+1/2}$$

Change of variables: useful theorems

Another choice of $\hat{\xi}$ and $\hat{\eta}$ that causes the variables to separate is

$$\hat{\xi} = \hat{x} \text{ and } \hat{\eta} = 0$$

so that $\hat{\eta}_1 = -\widehat{\dot{y}}$. The characteristic equations are

$$\frac{\mathrm{d}\hat{x}}{\hat{x}} = \frac{\mathrm{d}\hat{y}}{0} = \frac{\mathrm{d}\widehat{\dot{y}}}{\widehat{\dot{y}}}$$

whose integrals are $\hat{x}\widehat{\dot{y}}$ and $\hat{y}$. So a general form of the differential equation invariant to the group $(\hat{\xi}, \hat{\eta})$ is

$$\hat{x}\widehat{\dot{y}} = A(\hat{y}) \Rightarrow \frac{\widehat{\dot{y}}}{A(\hat{y})} = \hat{x}$$

which is separable.

Change of variables: useful theorems

Theorem 2. If we can solve

$$\xi F_x + \eta F_y = \hat{x} = F(x, y)$$
$$\xi G_x + \eta G_y = 0$$

then we can make a change of variables that makes the original ODE separable. Note that by definition G is a group invariant.

Theorem 3. If we have $X(x, y; \lambda) = \lambda x$ (the transformation of x is a stretching) then $\xi = x$. We can satisfy theorem 2 by setting $F = x$. For groups in which the transformation of x is a stretching, introducing a group invariant as a new variable in place of y and keeping x leads to a new ODE that is separable.

Exercise 8

Exercise 8. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ax + by + c.$$

Determine a group that leaves the equation invariant, and then find the general solution.

Homework

Murray, J.D., *Biological Mathematics: I. An Introduction*, Springer, New York, 2002.

Consider the following system that describes how two species of animals interact with each other and their environment

$$\dot{x} = x(3 - 2x - 2y) \text{ and } \dot{y} = y(2 - 2x - y)$$

(the rate of change of population is equal to the population weighted by the difference between birth and death rates).

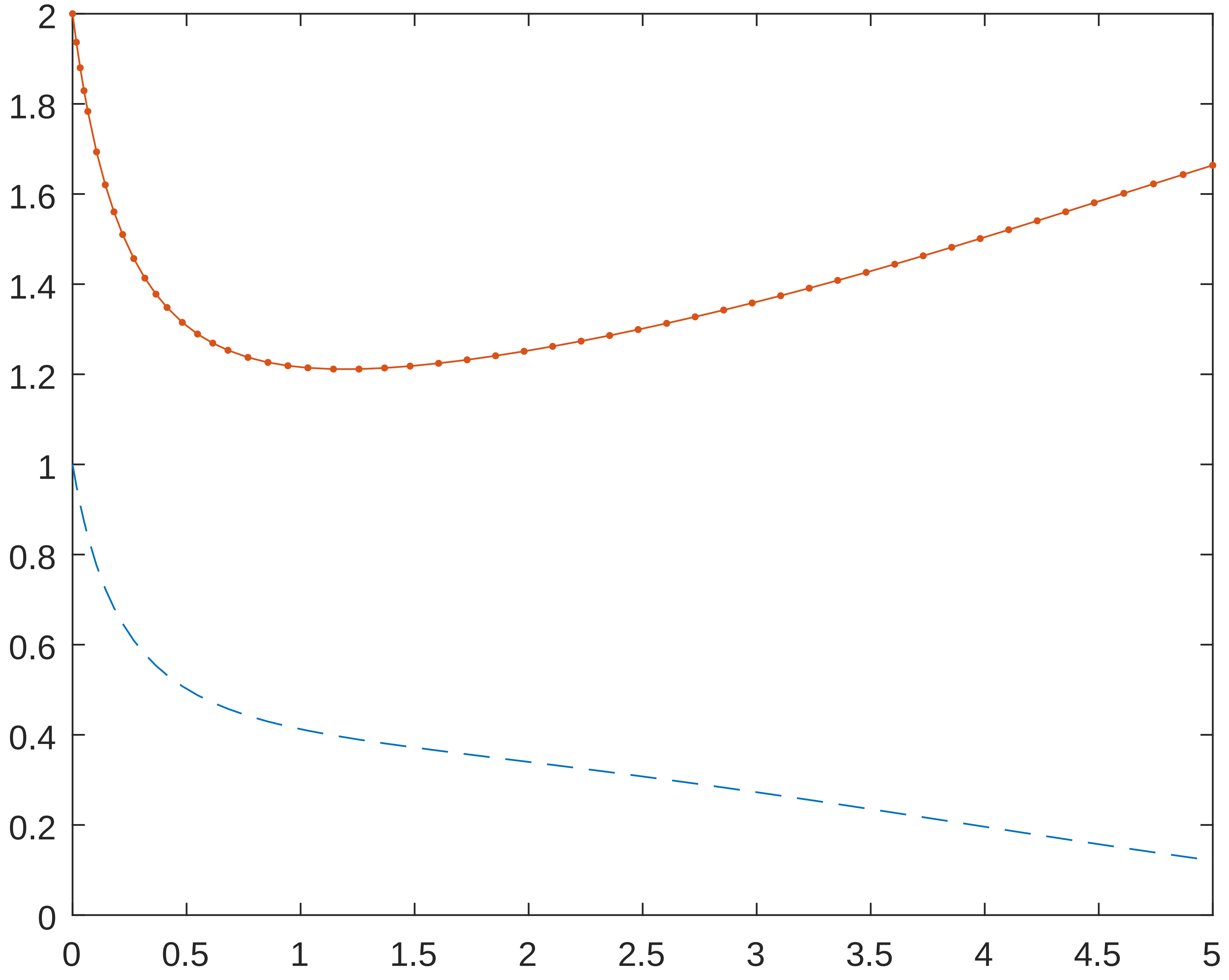
Homework

(i) Determine the critical points and the nullclines (the lines on which either $\dot{x} = 0$ or $\dot{y} = 0$). What is the behaviour of the solution at infinity?

(ii) Determine the separatrix.

(iii) Plot the phase portrait and the particular solution corresponding to $x(0) = 1$ and $y(0) = 2$

Homework: with Matlab



With Matlab: write a m-file

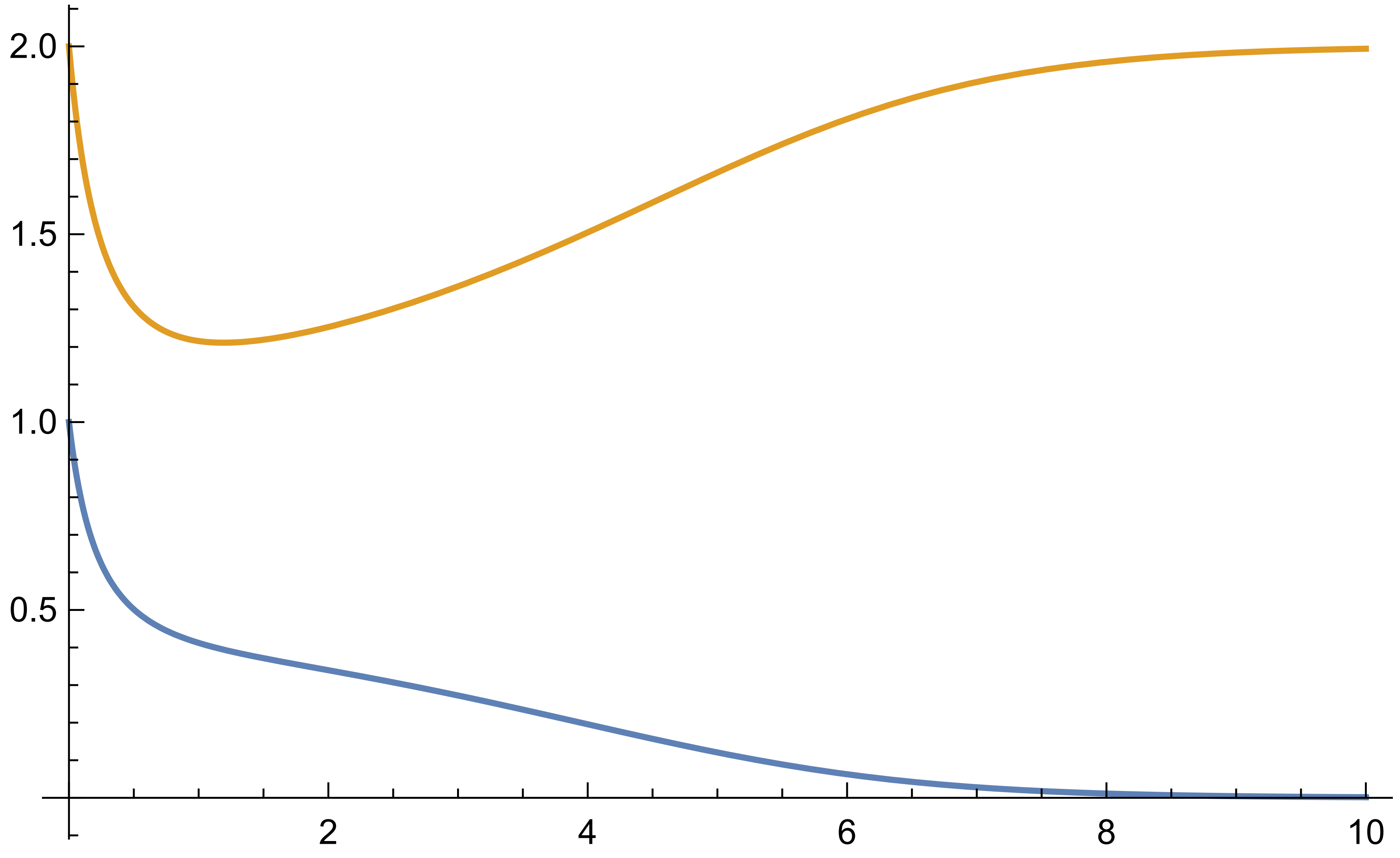
```
function dy=population (t,y)
dy(1) = y(1)*(3-2*y(1)-2*y(2));
dy(2) = y(2)*(2-2*y(1)-1*y(2));
dy=dy'
```

Then another script launches the computation and plots the solution

```
[t, y]=ode45(@population,[0 5],[1 2])
figure
plot(t,y(:,1),'–',t,y(:,2),'.-')
```

For the phase portrait, use *quiver*

Homework: with Mathematica



With Mathematica: use NDSolve[]

```
eqn = NDSolve[
{x'[t] == x[t] (3 - 2 x[t] - 2 y[t]),
y'[t] == y[t] (2 - 2 x[t] - y[t]),
x[0] == 1, y[0] == 2},
{x, y}, {t, 0, 10}]
des = Plot[{x[t] /. eqn, y[t] /. eqn}, {t, 0, 10}]
```

For the phase portrait, use *StreamPlot*